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Weighted Simultaneous Approximation by Lagrange Interpolation on the Real Line

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Abstract—In this paper, we discuss the simultaneous approximation of functions and their derivatives by Lagrange interpolating polynomials. We consider weighted approximation, where the weight is of Freud type. © 2004 Elsevier Ltd. All rights reserved.

Keywords—Weighted approximation, Lagrange interpolation, Freud type weight function.

We will discuss the simultaneous approximation of functions and their derivatives on the real line by Lagrange interpolating polynomials and their derivatives. We will estimate the speed of the convergence and give our error estimates in terms of the weighted norm of the interpolation operator.

Pointwise estimates on the real line for the error of simultaneous approximation of a function and its derivatives by Lagrange interpolation were given in Balázs [1,2]. On the positive half-line pointwise estimates were given in Balázs [3], and weighted pointwise estimates were given in Balázs and Kilgore [4].

When we consider weighted approximation on the real line, a natural choice for the weight function is an exponential weight of Freud type. The prototypes of the Freud-type weights are the weight functions $w(x) = e^{-\beta|x|^\alpha}$ for $\alpha > 1$ and $\beta > 0$, which include the important special case of the Hermite weight $e^{-x^2/2}$. From these prototypes, the definition of Freud-type weights has evolved over time. Finally, a unified treatment of Freud-type weights has appeared in Mhaskar [6].

Following Mhaskar's definition [6, p. 47], we will say that w is a *Freud-type weight* function if w can be represented as $w(x) = e^{-Q(x)}$, where Q satisfies the following conditions

- Q is an even, convex function on $\mathbf{R}$,
- Q is twice continuously differentiable on $(0, \infty)$,
- there are constants A and B such that

$$0 < A \le \frac{xQ''(x)}{Q'(x)} \le B < \infty.$$

If w is a Freud-type weight, then so is w^μ for all $\mu > 0$ [6, p. 81]. In particular, w^2 is also a Freud-type weight.

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The functions to be approximated will be from the space

$$C_w(\mathbf{R}) = \left\{ f : f \text{ continuous on } \mathbf{R} \text{ and } \lim_{|x|\to\infty} w(x)f(x) = 0 \right\},$$

and the spaces

$$C_w^k(\mathbf{R}) = \left\{ f : f^{(i)} \in C_w(\mathbf{R}), \text{ for } i = 0, 1, \dots, k \right\}.$$

For $f \in C_w(\mathbf{R})$, we have

$$E_n(f)_w = \inf_{p\in\prod_n} \|w(f-p)\|,$$

where $\prod_n$ denotes the set of polynomials of degree at most n, and the norm $\|\cdot\|$ denotes the usual supremum norm on the real line, $\mathbf{R}$.

To define Lagrange interpolation, let $X_n = \{x_1, x_2, \dots, x_n\}$ be a set of pairwise distinct nodes. Then, the Lagrange interpolation operator is defined by

$$L_n(f,x) = L(f, X_n, x) = \sum_{k=1}^{n} f(x_k)\ell_k(X_n, x),$$

where the functions $\ell_k(X_n, x)$ are the polynomials of degree at most $n-1$ which satisfy

$$\ell_k(X_n, x_j) = \delta_{kj}, \qquad \text{for } k, j = 1, \dots, n.$$

The good way to estimate the error of the approximation by interpolation (with the help of a P from $\prod_{n-1}$ for which $E_{n-1}(f)_w = \|w(f-P)\|$) is

$$\begin{aligned}\|w(x)(f(x) - L(f, X_n, x))\| &\le \|w(x)(f(x) - P(x))\| + \|w(x)(L(P-f), X_n, x)\| \\ &\le E_{n-1}(f)_w(1 + \|L(\cdot, X_n, \cdot)\|_w),\end{aligned}$$

a weighted version of Lebesgue's theorem, where $\|L(\cdot, X_n, \cdot)\|_w$ is defined as

$$\|L(\cdot, X_n, \cdot)\|_w = \left\| w(x) \sum_{k=1}^{n} (w(x_k))^{-1} |\ell_k(x)| \right\|,$$

which agrees with

$$\sup_{\|wf\|\le 1} \|w(x)L(f, X_n, x)\| = \sup_{\|f\|_w \le 1} \|L_n(f,x)\|_w = \|L_n\|_w,$$

the weighted norm of L_n defined in the standard manner. The norm $\|L(\cdot, X_n, \cdot)\|_w = \|L_n\|_w$ is also the *weighted Lebesgue constant*, $\lambda(X_n)_w$, associated with the nodes X_n.

The same weighted norm and the related weighted Lebesgue function for interpolation were used in Kilgore [5] to establish and prove the natural analogues of the Bernstein-Erdős conjectures about the optimal nodes of interpolation for polynomials with Hermite weights on $\mathbf{R}$ and with Laguerre weights on the half line.

Szabados [7] has shown that, in order to get the optimal order of magnitude for $\lambda(X_n)_w$, the optimal choice of nodes is not exactly the roots of orthogonal polynomials with weight w^2, although close. Szabados [7] constructed a system of nodes to achieve the optimal order of magnitude of $\lambda(X_n)_w$. His system was constructed by taking the roots

$$x_1 > x_2 > \cdots > x_n$$

of the polynomial P_n from the sequence of polynomials orthogonal with respect to w^2 and then adding two more nodes x_0 and $-x_0$. The node x_0 is defined in each case as a point where

$$|p_n(x_0)|w(x_0) = \|p_n w\|.$$

Szabados showed that $x_0 \neq 0$. He denoted this system of nodes by V_{n+2}, and he proved the following theorem.

THEOREM. *(See [7].) Let w be a Freud-type weight function and let*

$$V_{n+2} = \{x_1, x_2, \dots, x_n, x_0, -x_0\},$$

be the nodes of interpolation associated with w. Then,

$$\|L(\cdot, V_{n+2}, \cdot)\|_w = \lambda(V_{n+2})_w \sim \log n.$$

The symbol $\sim$ means that, the ratio of the two sides is between two positive constants.

COROLLARY. *(See [7].) Let w be a Freud-type weight and $f \in C_w(\mathbf{R})$. If*

$$E_n(f)_w \log n \to 0,$$

then,

$$\lim_{n\to\infty} \|w(x)(f(x) - L(f, V_{n+2}, x))\| = 0.$$

In our following theorem, we will show that simultaneous approximation of the derivatives also takes place.

The *Freud numbers* q_n will play an important role here. They will be defined as in Mhaskar [6, p. 81, formula 4.1.2]. Let $w(x) = e^{-Q(x)}$ be a Freud-type weight. Then, for every integer n, q_n is the (unique) positive number such that $q_n Q'(q_n) = n$. As to the behavior of the numbers q_n, it follows from their definition that q_n increases with n and that q_n/n is decreasing, with

$$\lim_{n\to\infty} \frac{q_n}{n} = 0. \tag{1}$$

Also, (see, for example [6, (4.1.3)], second line), there are positive constants c_1, c_2, c_3, c_4 such that

$$c_1 n^{1/(1+c_2)} \leq q_n \leq c_3 n^{1/(1+c_4)}. \tag{2}$$

In the following theorem, we give weighted error estimates for the approximation of a function and its derivatives.

THEOREM. *Let w be a Freud-type weight and $f \in C_w^k(\mathbf{R})$. Then, for $k < n$ and for $i = 0, 1, \dots, k$*

$$\left\| w(x) \left(f^{(i)}(x) - L^{(i)}(f, X_{n+1}, x) \right) \right\| \leq \alpha_{i,k} \left(\frac{q_{n-k}}{n-k} \right)^{k-i} E_{n-k}\left(f^{(k)} \right)_w \left(1 + \|L(\cdot, X_{n+1}, \cdot)\|_w\right),$$

where $\alpha_{i,k}$ is a constant depending on i, on k, and on w only.

If we choose for nodes the set V_{n+2}, we can give concrete estimates for the weighted simultaneous approximation of the derivatives of a function.

COROLLARY 1. *Let w be a Freud-type weight and $f \in C_w^k(\mathbf{R})$. Then, for interpolation on the nodes V_{n+2} we have for $i = 0, 1, \dots, k$*

$$\left\| w(x) \left(f^{(i)}(x) - L^{(i)}(f, V_{n+2}, x) \right) \right\| \leq \alpha_{i,k} \left(\frac{q_{n+1-k}}{n+1-k} \right)^{k-i} E_{n-k+1}\left(f^{(k)} \right)_w \log n,$$

where $\alpha_{i,k}$ is a constant depending on i, on k, and on w only.

Taking into account the rate of growth of the numbers q_n given in (1) and (2), a Dini-Lipschitz type convergence theorem for simultaneous approximation also follows.

COROLLARY 2. *Let w be a Freud-type weight and $f \in C_w^k(\mathbf{R})$, and let the nodes of interpolation be V_{n+2}. Then, for each $i = 0, 1, \ldots, k$, if*

$$\left(\frac{q_{n+1-k}}{n+1-k}\right)^{k-i} E_n\left(f^{(k)}\right)_w \log n \to 0,$$

then,

$$\lim_{n\to\infty} \left\| w(x)\left(f^{(i)}(x) - L^{(i)}\left(f, V_{n+2}, x\right)\right)\right\| = 0.$$

The theorem and its corollaries will be proved with the help of the following lemmas.

LEMMA 1. *Markov-Bernstein-type inequality. (See [6, Theorem 6.2.9.) Let $w(x) = e^{-Q(x)}$ be a Freud-type weight. Then, there is a positive constant c depending only on w such that for every integer $n \geq 1$ and every $P \in \prod_n$*

$$\|w(x)P'(x)\| \leq c\frac{n}{q_n}\|w(x)P(x)\|.$$

LEMMA 2. *Let w be a Freud-type weight and $f \in C_w^1(\mathbf{R})$. For an integer $n \geq 1$ and for $P \in \prod_n$, if for some $c \geq 1$*

$$\|w(x)(f(x) - P(x))\| \leq cE_n(f)_w,$$

then, with c_1 depending only on c and w,

$$\|w(x)\left(f'(x) - P'(x)\right)\| \leq c_1 E_{n-1}\left(f'\right)_w.$$

PROOF. This lemma is a special case of Theorem 4.1.7 of [6], where it is assumed that a Markov-Bernstein-type inequality (as stated in Lemma 1) is satisfied. ∎

LEMMA 3. *Let w be a Freud-type weight and $f \in C_w^1(\mathbf{R})$. Then, there is a constant c depending only upon w such that for every integer $n \geq 1$,*

$$E_n(f)_w \leq c\frac{q_n}{n}E_{n-1}(f')_w.$$

PROOF. This lemma is a special case of Theorem 4.1.1 of [6], formula (4.1.5b), where it is assumed that a Markov-Bernstein-type inequality is satisfied, as stated in Lemma 1. ∎

PROOF of the Theorem. Let P be a polynomial of degree at most n for which

$$\|w(x)(f(x) - P(x))\| = E_n(f)_w.$$

Then, using the fact that $P^{(i)}(x) = L(P^{(i)}, X_{n+1}, x) = L^{(i)}(P, X_{n+1}, x)$, we have

$$\begin{aligned}\left\|w(x)\left(f^{(i)}(x) - L^{(i)}\left(f, X_{n+1}, x\right)\right)\right\| &\leq \left\|w(x)\left(f^{(i)}(x) - P^{(i)}(x)\right)\right\| \\ &\quad + \left\|w(x)\left(L\left(P^{(i)}, X_{n+1}, x\right) - L^{(i)}\left(f, X_{n+1}, x\right)\right)\right\| \\ &= \left\|w(x)\left(f^{(i)}(x) - P^{(i)}(x)\right)\right\| + \left\|w(x)L^{(i)}\left(f - P, X_{n+1}, x\right)\right\|.\end{aligned}$$

By repeated application of Lemma 2, the first term on the right can be estimated as

$$\left\|w(x)\left(f^{(i)}(x) - P^{(i)}(x)\right)\right\| \leq c_i E_{n-i}\left(f^{(i)}\right)_w,$$

in which c_i is a constant depending only on i (and w).

The second term on the right can be estimated by repeatedly applying Lemmas 1 and 3 in alternation, obtaining

$$\left\| w(x) L^{(i)}\left(f-P, X_{n+1}, x\right)\right\| \leq \gamma_i E_{n-i}\left(f^{(i)}\right)_w \left\|L\left(\cdot, X_{n+1}, \cdot\right)\right\|_w,$$

with γ_i depending only on i and w. Therefore,

$$\begin{aligned}\left\| w(x)\left(f^{(i)}(x)-L^{(i)}\left(f, X_{n+1}, x\right)\right)\right\| &\leq \left\| w(x)\left(f^{(i)}(x)-P^{(i)}(x)\right)\right\| \\ &\quad + \left\| w(x) L^{(i)}\left(f-P, X_{n+1}, x\right)\right\| \\ &\leq c_i E_{n-i}\left(f^{(i)}\right)_w + \gamma_i E_{n-i}\left(f^{(i)}\right)_w \left\|L\left(\cdot, X_{n+1}, \cdot\right)\right\|_w \\ &\leq \max\{c_i, \gamma_i\} E_{n-i}\left(f^{(i)}\right)_w \left(1+\left\|L\left(\cdot, X_{n+1}, \cdot\right)\right\|_w\right).\end{aligned}$$

Noting that, when $i \leq j \leq k-1$, we have

$$\frac{q_{n-j}}{n-j} < \frac{q_{n-k}}{n-k},$$

by repeated use of Lemma 3, we obtain

$$\left\| w(x)\left(f^{(i)}(x)-L^{(i)}\left(f, X_{n+1}, x\right)\right)\right\| \leq \alpha_{i,k}\left(\frac{q_{n-k}}{n-k}\right)^{k-i} E_{n-k}\left(f^{(k)}\right)_w \left(1+\left\|L\left(\cdot, X_{n+1}, \cdot\right)\right\|_w\right),$$

which concludes the proof of the theorem. ∎

PROOF of Corollary 1. This follows immediately, by applying in our theorem the estimate

$$\|L(\cdot, V_{n+2}, \cdot)\|_w = \lambda(V_{n+2})_w \sim \log n,$$

from the theorem of Szabados quoted above. ∎

PROOF of Corollary 2. Corollary 2 is an immediate consequence of Corollary 1. ∎

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